Odchýlka dvoch vektorov

Odchýlka (uhol) dvoch vektorov je konvexný uhol z intervalu $\varphi \in \langle 0°; 180°\rangle$. Najmenší uhol je nula stupňový uhol – prípad, keď vektory majú rovnaký smer. Najväčší uhol je 180 stupňový – keď vektory majú opačný smer. Takže nekončí uhol pri 90° stupňoch, ako pri priamkach.
Ak upravíme vzťah z definície skalárneho súčinu vektorov (súčin veľkostí vektorov násobený s kosínusom nimi zovretého uhla), a vyjadríme uhol (vlastne kosínus toho uhla), dostaneme vzťah, ktorým môžeme ľahko vypočítať odchýlku.

$$\vec{a}.\vec{b} = |\vec{a}|.|\vec{b}|.\cos\varphi \qquad /:(|\vec{a}|.|\vec{b}|)$$

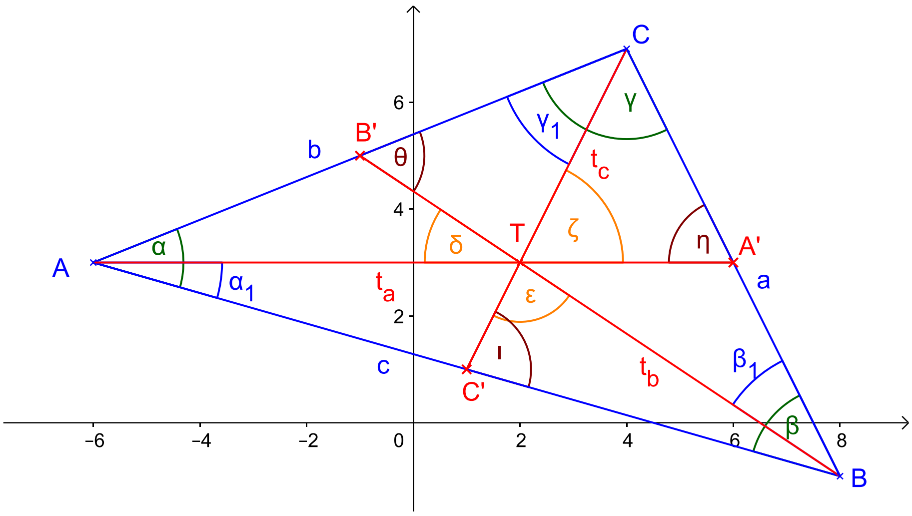
$$\frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|} = \cos\varphi$$

V. Ak sú dané vektory $\vec{a} = (a_1; a_2)$; $\vec{b} = (b_1; b_2)$, potom ich odchýlka:

$$\cos\varphi = \frac{\vec{a}.\vec{b}}{|\vec{a}|.|\vec{b}|} = \frac{a_1.b_1+a_2.b_2}{\sqrt{a_1^2+a_2^2}.\sqrt{b_1^2+b_2^2}}$$

príklad:

Daný je trojuholník ABC vrcholmi: A(-6; 3), B(8; -1), C(4; 7). Vypočítajte jeho vnútorné uhly (α, β, γ), uhly ťažníc (δ, ε, ζ), ťažníc a strán (η, θ, ι; α_1, β_1, γ_1).



$\alpha = \sphericalangle b, c = \sphericalangle \overrightarrow{AB}, \overrightarrow{AC}$

$\overrightarrow{AB} = B - A = (14; -4) \sim (7; -2)$ $\qquad$ $\overrightarrow{AC} = C - A = (10; 4) \sim (5; 2)$

$$\cos\alpha = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{|\overrightarrow{AB}|.|\overrightarrow{AC}|} = \frac{7.5+(-2).2}{\sqrt{7^2+(-2)^2}.\sqrt{5^2+2^2}} = \frac{35-4}{\sqrt{49+4}.\sqrt{25+4}} = \frac{31}{\sqrt{53.29}} = 0{,}790\ 7$$

$\alpha = \cos^{-1} 0{,}790\ 7$

$\alpha = 37°44'48''$

$\beta = \sphericalangle a, c = \sphericalangle \overrightarrow{BC}, \overrightarrow{BA}$

$\overrightarrow{BC} = C - B = (-4; 8) \sim (-1; 2)$ $\qquad$ $\overrightarrow{BA} = -\overrightarrow{AB} = (-7; 2)$

$$\cos\beta = \frac{\overrightarrow{BC}.\overrightarrow{BA}}{|\overrightarrow{BC}|.|\overrightarrow{BA}|} = \frac{-1.(-7)+2.2}{\sqrt{(-1)^2+2^2}.\sqrt{(-7)^2+2^2}} = \frac{7+4}{\sqrt{1+4}.\sqrt{49+4}} = \frac{11}{\sqrt{5.53}} = 0{,}675\ 7$$

$\beta = \cos^{-1} 0{,}675\ 7$

$\beta = 47°29'22''$

$\gamma = \sphericalangle a, b = \sphericalangle \overrightarrow{CA}, \overrightarrow{CB}$

$\overrightarrow{CA} = -\overrightarrow{AC} = (-5; -2)$ $\qquad$ $\overrightarrow{CB} = -\overrightarrow{BC} = (1; -2)$

$$\cos\gamma = \frac{\overrightarrow{CA}.\overrightarrow{CB}}{|\overrightarrow{CA}|.|\overrightarrow{CB}|} = \frac{-5.1+(-2).(-2)}{\sqrt{(-5)^2+(-2)^2}.\sqrt{1^2+(-2)^2}} = \frac{-5+4}{\sqrt{25+4}.\sqrt{1+4}} = \frac{-1}{\sqrt{29.5}} = -0{,}083\ 0$$

$\gamma = \cos^{-1}(-0{,}083\ 0)$

$\gamma = 94°45'49''$

$\delta = \sphericalangle t_a, t_b = \sphericalangle \overrightarrow{AA'}, \overrightarrow{BB'}$

$A' = \frac{B+C}{2} = (6; 3)$ $\qquad$ $B' = \frac{A+C}{2} = (-1; 5)$

$\overrightarrow{AA'} = A' - A = (12; 0) \sim (1; 0)$ $\overrightarrow{BB'} = B' - B = (-9; 6) \sim (-3; 2)$

$\cos\delta = \frac{\overrightarrow{AA'}.\overrightarrow{BB'}}{|\overrightarrow{AA'}|.|\overrightarrow{BB'}|} = \frac{1.(-3)+0.2}{\sqrt{1^2+0^2}.\sqrt{(-3)^2+2^2}} = \frac{-3+0}{\sqrt{1+0}.\sqrt{9+4}} = \frac{-3}{\sqrt{1.13}} = -0{,}832\ 1$

$\delta' = \cos^{-1}(-0{,}832\ 1) = 146°18'36''$ $\Rightarrow$ $\delta = 180° - \delta'$

$\delta = 33°41'24''$

$\varepsilon = \sphericalangle\ t_b, t_c = \sphericalangle\ \overrightarrow{BB'}, \overrightarrow{CC'}$

$C' = \frac{A+B}{2} = (1; 1)$

$\overrightarrow{BB'} = (-3; 2)$ $\overrightarrow{CC'} = C' - C = (-3; -6) \sim (-1; -2)$

$\cos\varepsilon = \frac{\overrightarrow{BB'}.\overrightarrow{CC'}}{|\overrightarrow{BB'}|.|\overrightarrow{CC'}|} = \frac{-3.(-1)+2.(-2)}{\sqrt{(-3)^2+2^2}.\sqrt{(-1)^2+(-2)^2}} = \frac{3-4}{\sqrt{9+4}.\sqrt{1+4}} = \frac{-1}{\sqrt{13.5}} = -0{,}124\ 0$

$\varepsilon' = \cos^{-1}(-0{,}124\ 0) = 97°7'30''$ $\Rightarrow$ $\varepsilon = 180° - \varepsilon'$

$\varepsilon = 82°52'30''$

$\zeta = \sphericalangle\ t_c, t_a = \sphericalangle\ \overrightarrow{CC'}, \overrightarrow{AA'}$

$\overrightarrow{CC'} = (-1; -2)$ $\overrightarrow{AA'} = (1; 0)$

$\cos\zeta = \frac{\overrightarrow{CC'}.\overrightarrow{AA'}}{|\overrightarrow{CC'}|.|\overrightarrow{AA'}|} = \frac{-1.1+(-2).0}{\sqrt{(-1)^2+(-2)^2}.\sqrt{1^2+0^2}} = \frac{-1+0}{\sqrt{1+4}.\sqrt{1+0}} = \frac{-1}{\sqrt{5.1}} = -0{,}447\ 2$

$\zeta' = \cos^{-1}(-0{,}447\ 2) = 116°33'54''$ $\Rightarrow$ $\zeta = 180° - \zeta'$

$\zeta = 63°26'6''$

$\eta = \sphericalangle\ a, t_a = \sphericalangle\ \overrightarrow{BC}, \overrightarrow{A'A}$

$\overrightarrow{BC} = (-1; 2)$ $\overrightarrow{A'A} = -\overrightarrow{AA'} = (-1; 0)$

$\cos\eta = \frac{\overrightarrow{BC}.\overrightarrow{A'A}}{|\overrightarrow{BC}|.|\overrightarrow{A'A}|} = \frac{-1.(-1)+2.0}{\sqrt{(-1)^2+2^2}.\sqrt{(-1)^2+0^2}} = \frac{1+0}{\sqrt{1+4}.\sqrt{1+0}} = \frac{1}{\sqrt{5.1}} = 0{,}447\ 2$

$\eta = \cos^{-1} 0{,}447\ 2$

$\eta = 63°26'6''$

$\theta = \sphericalangle\ b, t_b = \sphericalangle\ \overrightarrow{AC}, \overrightarrow{B'B}$

$\overrightarrow{AC} = (5; 2)$ $\overrightarrow{B'B} = -\overrightarrow{BB'} = (3; -2)$

$\cos\theta = \frac{\overrightarrow{AC}.\overrightarrow{B'B}}{|\overrightarrow{AC}|.|\overrightarrow{B'B}|} = \frac{5.3+2.(-2)}{\sqrt{5^2+2^2}.\sqrt{3^2+(-2)^2}} = \frac{15-4}{\sqrt{25+4}.\sqrt{9+4}} = \frac{11}{\sqrt{29.13}} = 0{,}566\ 5$

$\theta = \cos^{-1} 0{,}566\ 5$

$\theta = 55°29'29''$

$\iota = \sphericalangle\ c, t_c = \sphericalangle\ \overrightarrow{AB}, \overrightarrow{C'C}$

$\overrightarrow{AB} = (7; -2)$ $\overrightarrow{C'C} = -\overrightarrow{CC'} = (1; 2)$

$\cos\iota = \frac{\overrightarrow{AB}.\overrightarrow{C'C}}{|\overrightarrow{AB}|.|\overrightarrow{C'C}|} = \frac{7.1+(-2).2}{\sqrt{7^2+(-2)^2}.\sqrt{1^2+2^2}} = \frac{7-4}{\sqrt{49+4}.\sqrt{1+4}} = \frac{3}{\sqrt{53.5}} = 0{,}184\ 3$

$\iota = \cos^{-1} 0{,}184\ 3$

$\iota = 79°22'49''$

$\alpha_1 = \sphericalangle\ c, t_a = \sphericalangle\ \overrightarrow{AB}, \overrightarrow{AA'}$

$\overrightarrow{AB} = (7; -2)$ $\overrightarrow{AA'} = (1; 0)$

$\cos\alpha_1 = \frac{\overrightarrow{AB}.\overrightarrow{AA'}}{|\overrightarrow{AB}|.|\overrightarrow{AA'}|} = \frac{7.1+(-2).0}{\sqrt{7^2+(-2)^2}.\sqrt{1^2+0^2}} = \frac{7+0}{\sqrt{49+4}.\sqrt{1+0}} = \frac{7}{\sqrt{53.1}} = 0{,}961\ 5$

$\alpha_1 = \cos^{-1} 0{,}961\ 5$

$\alpha_1 = 15°56'43''$

$\beta_1 = \sphericalangle\ a, t_b = \sphericalangle\ \overrightarrow{BC}, \overrightarrow{BB'}$

$\overrightarrow{BC} = (-1; 2)$ $\overrightarrow{BB'} = (-3; 2)$

$\cos\beta_1 = \frac{\overrightarrow{BC}.\overrightarrow{BB'}}{|\overrightarrow{BC}|.|\overrightarrow{BB'}|} = \frac{-1.(-3)+2.2}{\sqrt{(-1)^2+2^2}.\sqrt{(-3)^2+2^2}} = \frac{3+4}{\sqrt{1+4}.\sqrt{9+4}} = \frac{7}{\sqrt{5.13}} = 0{,}868\ 2$

$\beta_1 = \cos^{-1} 0{,}868\,2$

$\beta_1 = 29°44'42''$

$\gamma_1 = \sphericalangle\, b, t_c = \sphericalangle\, \overrightarrow{CA}, \overrightarrow{CC'}$

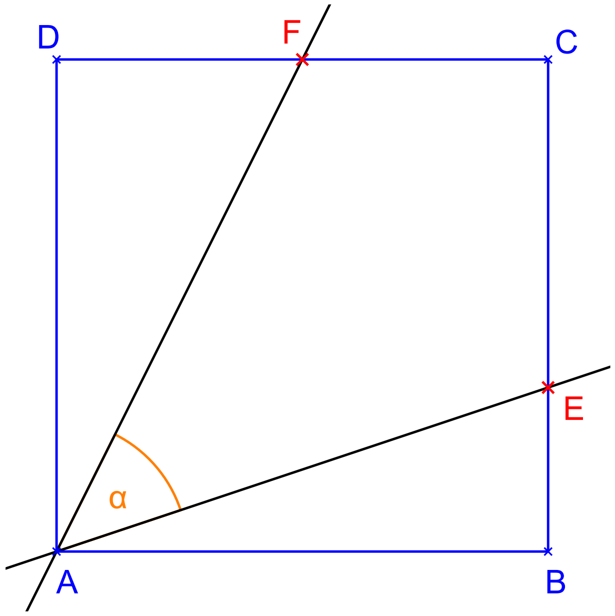
$\overrightarrow{CA} = (-5; -2)$ $\overrightarrow{CC'} = (-1; -2)$

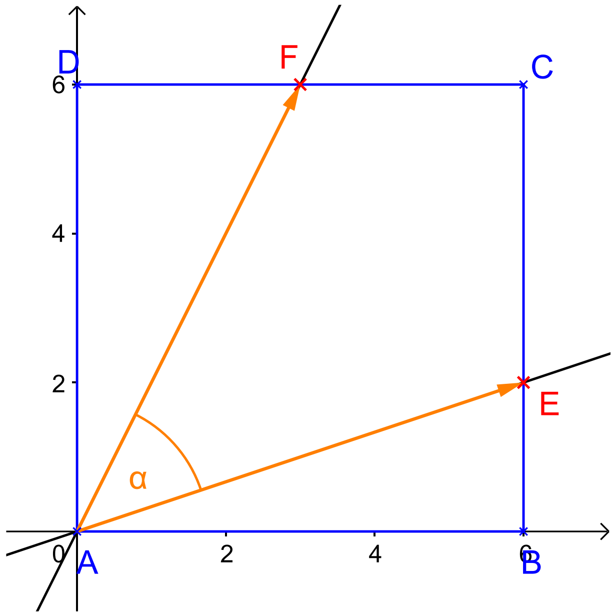
$$\cos \gamma_1 = \frac{\overrightarrow{CA}.\overrightarrow{CC'}}{|\overrightarrow{CA}|.|\overrightarrow{CC'}|} = \frac{-5.(-1)+(-2).(-2)}{\sqrt{(-5)^2+(-2)^2}.\sqrt{(-1)^2+(-2)^2}} = \frac{5+4}{\sqrt{25+4}.\sqrt{1+4}} = \frac{9}{\sqrt{29.5}} = 0{,}747\,4$$

$\gamma_1 = \cos^{-1} 0{,}747\,4$

$\gamma_1 = 41°38'1''$

Z vrcholu A štvorca ABCD sú vedené priamky, ktoré prechádzajú bodmi E a F. Bod E leží v tretine strany BC, bližšie k vrcholu B. Bod F rozpoľuje stranu CD. Nájdite uhol, ktorý tieto priamky zvierajú.





umiestnime štvorec do súradnicovej sústavy – keďže jeden bod je v polovici, druhý v tretine, stranu si zvolíme na 6 jednotiek ⇒ E = (6; 2), F = (3; 6)

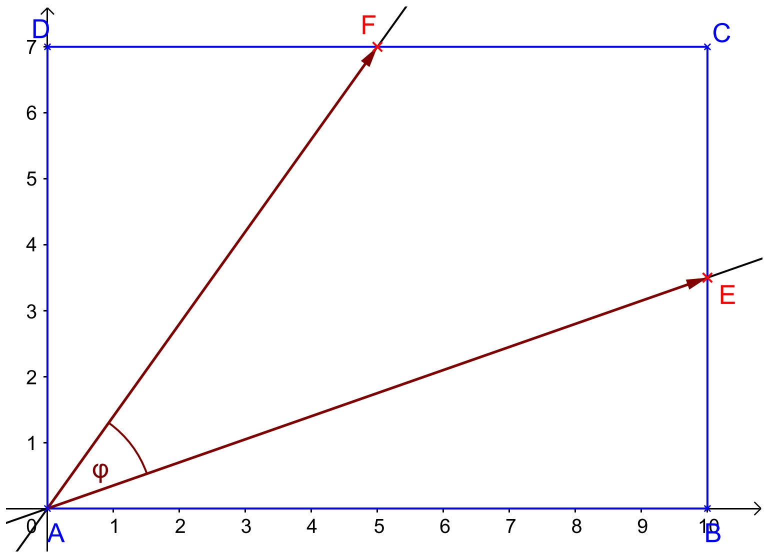
$\overrightarrow{AE} = E - A = (6; 2) \sim (3; 1)$ $\overrightarrow{AF} = F - A = (3; 6) \sim (1; 2)$

$$\cos \alpha = \frac{\overrightarrow{AE}.\overrightarrow{AF}}{|\overrightarrow{AE}|.|\overrightarrow{AF}|} = \frac{3.1+1.2}{\sqrt{3^2+1^2}.\sqrt{1^2+2^2}} = \frac{3+2}{\sqrt{9+1}.\sqrt{1+4}} = \frac{5}{\sqrt{10.5}} = 0{,}707\,1$$

$\alpha = \cos^{-1} 0{,}707\,1$

$\alpha = 45°$

Z jedného vrcholu obdĺžnika s rozmermi 10×7 sú vedené dve priamky, ktoré rozpoľujú protiľahlé strany. Aký uhol zvierajú tieto priamky?



$\overrightarrow{AE} = E - A = (10; 3{,}5)$ $\overrightarrow{AF} = F - A = (5; 7)$

$\cos \varphi = \frac{\overrightarrow{AE}.\overrightarrow{AF}}{|\overrightarrow{AE}|.|\overrightarrow{AF}|} = \frac{10.5+3,5.7}{\sqrt{10^2+3,5^2}.\sqrt{5^2+7^2}} = \frac{50+24,5}{\sqrt{100+12,25}.\sqrt{25+49}} = \frac{74,5}{\sqrt{112,25.74}} = 0,817\ 4$

$\varphi = \cos^{-1} 0,817\ 5$

$\varphi = 35°10'20''$